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# Control of Flexible Spacecraft with Time-Varying Configuration

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**This paper is concerned with the dynamics and control of systems with time-varying configuration, such as maneuvering articulated flexible spacecraft. The mathematical model consists of a rigid platform and a given number of retargeting flexible antennas. The mission consists of maneuvering the antennas so as to coincide with preselected lines of sight while stabilizing the platform in an inertial space and suppressing the elastic vibration of the antennas. A perturbation technique permits the derivation of a new control law for systems with time-varying configuration, in which the time-varying terms are relatively small. According to the proposed perturbation method, the control gains consist of zero-order time-invariant gains obtained from the solution of a matrix algebraic Riccati equation, which are valid both during and after the maneuver, and higher-order time-varying gains obtained from the solution of matrix differential Lyapunov equations, which are valid only during the maneuver. The approach is illustrated by means of a numerical example.**

## Introduction

THE interest lies in the maneuvering of articulated flexible spacecraft (Fig. 1). In particular, the maneuver involves the reorientation relative to an inertial space of a line of sight embedded in a flexible component of the spacecraft, such as a flexible antenna. To this end, it is convenient to adopt a strategy in which the main body of the spacecraft, regarded as a rigid platform, is stabilized relative to the inertial space and the flexible component is maneuvered relative to the platform so as to cause the line of sight to coincide with the desired direction in space. Note that such a maneuvering spacecraft is characterized by the fact that its configuration varies with time.

The control task can be designed by first designing the maneuvering of the antenna as if it were rigid. Of course, in actuality, the antenna is not rigid, so that the maneuver is likely to cause elastic vibration, which in turn will induce perturbations in the motion of the whole spacecraft. Hence, in addition to the rigid-body maneuvering, the control task amounts to simultaneous stabilization of the rigid platform relative to an inertial space and suppression of the elastic vibration caused by the maneuvering.

The rigid-body maneuvering of the antenna can be carried out open loop. In the case of a minimum-time maneuver, the control law is bang-bang. On the other hand, the control of the elastic vibration and of the perturbations about the rigid-body maneuver caused by the elastic vibration is carried out closed loop. Note that the motion defining the open-loop rigid-body maneuver can be regarded as a known function of time. As a result, the state equations for feedback control are characterized by time-varying coefficients and persistent disturbances, both arising from the known rigid-body maneuver.

The problem just described has been treated previously[1] under the assumption that the maneuver is slow, which implies that the time-varying terms in the coefficients are relatively small. The controller used in Ref. 1 consists of a disturbance-accommodating control designed to counteract the effect of the persistent disturbances and constant-gains feedback controls designed on the basis of the premaneuver configuration.

Bang-bang control implies that the maneuver angular acceleration is constant over the first half of the maneuver, reverses sign at one half of the maneuver period, and continues at the same level over the second half of the maneuver. Proportional-plus-integral (PI) control proved to be effective in the case of constant disturbances.[2,3] This approach was used in Ref. 4 to control the vibration and the rigid-body perturbations in a spacecraft with two flexible antenna undergoing simultaneous maneuvering. Under the same assumption as in Ref. 1 that the time-varying part in the coefficients is small, the control design was based in Ref. 4 on the constant part, which implies the use of constant gains.

This paper presents a different approach to the problem than in Refs. 1 and 4. The approach amounts to assuming that the control gains consist of a large constant part and a small time-varying part. This permits the use of a perturbation approach, whereby the solution of the matrix Riccati equation is divided into a zero-order (in the perturbation sense) problem requiring the solution of a matrix differential Riccati equation for the constant part of the Riccati matrix and higher-order problems amounting to the solution of matrix differential Lyapunov equations for the time-varying part.

Another approach to the control of time-varying systems is referred to as an adiabatic approximation.[5] Implicit in the adiabatic approximation is not only that the time-varying part

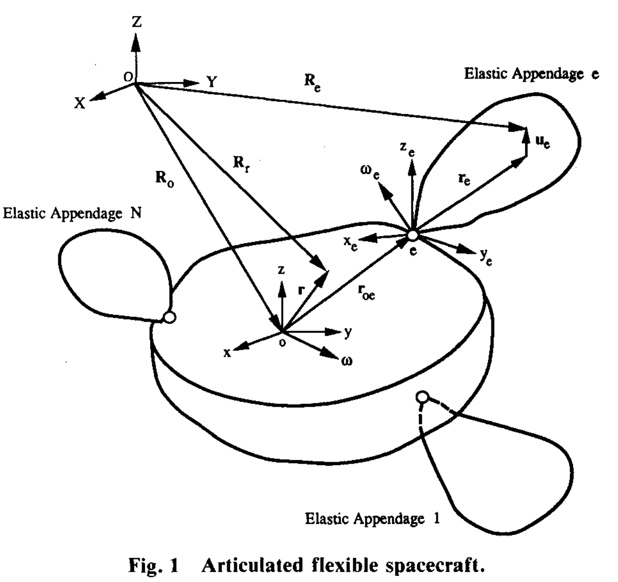


Fig. 1 Articulated flexible spacecraft.

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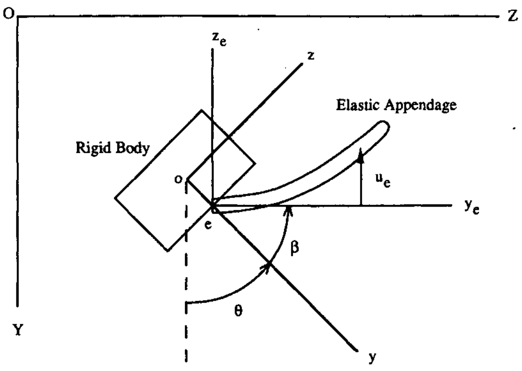


**Fig. 2 Flexible spacecraft in planar motion.**

in the Riccati matrix is small but also that the time variation is slow.

The developments presented here are illustrated by means of an example involving the planar motion of a spacecraft consisting of a rigid platform and a maneuvering flexible appendage (Fig. 2).

## Equations of Motion

In this section, the equations of motion for a spacecraft with retargeting flexible antennas are presented. The spacecraft is assumed to consist of a rigid body and several flexible antennas, where the flexible appendages are regarded as distributed parameter members.

The equations describing the rigid-body motions of the spacecraft are nonlinear ordinary differential equations. On the other hand, the equations describing the small elastic displacements of a flexible appendage relative to a frame embedded in the undeformed appendage are partial differential equations. Hence, the complete equations describing a spacecraft during reorientation represent a set of nonlinear hybrid differential equations.

In general, hybrid systems of equations do not permit closed-form solution, so that one must consider an approximate solution, which implies spatial discretization and truncation. Spatial discretization and truncation can be carried out by representing the motion as a finite set of admissible functions multiplied by time-dependent generalized coordinates. Moreover, the equations can be first linearized and then recast in compact state form. The state equations contain time-varying coefficients and persistent disturbances.

Let us consider the motion of a system consisting of a main rigid body and several flexible appendages hinged to the main body (Fig. 1). The equations of motion for a system of the type shown in Fig. 1 were derived from Ref. 1 and, for brevity, will not be repeated here in full. For small rigid-body and elastic motions, the equations can be reduced to the form

$$[M_0 + M_1(t)]\ddot{x} + G_1(t)\dot{x} + [K_0 + K_1(t)]x = f(t) + d(t) \quad (1)$$

where $x = [R^T\ \theta^T\ q_e^T]^T$ is the configuration vector in which $R$ represents the translational displacement vector of the platform body axes, $\theta$ represents the angular displacement vector of the same body axes, and $q_e$ is a vector of generalized coordinates corresponding to the elastic displacements of a typical flexible appendage $e$. Moreover, $M_0$ and $K_0$ are, respectively, symmetric mass and stiffness matrices corresponding to the spacecraft in the premaneuver configuration, and $M_1(t)$, $G_1(t)$, and $K_1(t)$ are time-varying matrices depending on the prescribed angular displacement, velocity, and acceleration, respectively, of the flexible appendage relative to the platform. Finally, $f(t)$ and $d(t)$ denote control and disturbance force vectors, respectively, where the latter represents inertial forces due to the maneuver.

The equations of motion can be simplified to some extent by expanding the configuration vector in a series of eigenvectors corresponding to the premaneuver spacecraft. To this end, we consider the eigenvalue problem

$$K_0 U = M_0 U \Lambda_0 \quad (2)$$

where $\Lambda_0 = \text{diag}(\lambda_i)$ is the matrix of eigenvalues and $U = [u_1, u_2, \ldots, u_n]$ is the matrix of eigenvectors. Because $K_0$ and $M_0$ are symmetric, the eigenvectors are orthogonal with respect to both $M_0$ and $K_0$. Moreover, $M_0$ is positive definite and $K_0$ is positive semidefinite, so that the eigenvalues are non-negative. The eigenvectors can be normalized so that the matrix $U$ satisfies

$$U^T M_0 U = I \quad (3a)$$

$$U^T K_0 U = \Lambda_0 \quad (3b)$$

where $I$ is the identity matrix. Then, introducing the linear transformation

$$x = U\eta \quad (4)$$

into Eq. (1), multiplying on the left by $U^T$ and considering Eqs. (3), we obtain

$$(I + \overline{M_1})\ddot{\eta} + \overline{G_1}\dot{\eta} + (\Lambda_0 + \overline{K_1})\eta = F + D \quad (5)$$

where

$$\overline{M_1} = U^T M_1 U \quad (6a)$$

$$\overline{G_1} = U^T G_1 U \quad (6b)$$

$$\overline{K_1} = U^T K_1 U \quad (6c)$$

$$F = U^T f \quad (6d)$$

$$D = U^T d \quad (6e)$$

Equation (5) is said to be in pseudomodal form.

## Maneuvering and Disturbances

The maneuver under consideration consists of retargeting the antennas so as to point in given directions in the inertial space. By stabilizing the platform in an inertial space, the task reduces to reorienting the antennas relative to the platform. For a minimum-time maneuver, the control law is bang-bang, which implies that the angular acceleration of an antenna relative to the platform is constant, with the sign changing at half the maneuver period. Ideally, the maneuver should not cause elastic deformations in the flexible appendages, which is not possible in theory. Hence, elastic deformations are likely to occur, which in turn implies perturbations of the platform from a fixed position in the inertial space. To suppress the elastic vibration and the perturbation of the platform, we propose to use feedback control.

The system governed by Eq. (1), or Eq. (5), is characterized by two factors that distinguish it from most commonly encountered systems: it is time varying and it is subjected to persistent disturbances. Both factors arise from the retargeting maneuver angular velocities $\omega_e$, angular acceleration $\dot{\omega}_e$, and the matrices $E_e$ of direction cosines between axes $x_e y_e z_e$ and axes $xyz$ ($e = 1, 2, \ldots, N$), all quantities being prescribed functions of time.

The persistent disturbances considered here take place during the maneuver and arise from known sources. Indeed, these disturbances arise from the inertial loading due to the motion of the flexible appendages, and they depend on the policy of reorientation of the flexible appendages. Disturbances tend to have undesirable effects on the pointing accuracy of spacecraft. As a result, it is necessary to design counteracting controls to mitigate any adverse effects. Moreover, discretization and truncation of the distributed-parameter system result in reduced-order realization for the disturbances. Hence, the a priori information concerning $d(t)$ is usually not sufficiently complete to permit accurate description of the nature of the disturbances, so that we are faced with the problem of designing disturbance-accommodating control with only an incomplete knowledge of these disturbances.

In this paper, we consider first a disturbance-minimizing control. To cope with incompletely known disturbances, PI control is explored. Using a perturbation method, the PI control is extended to the problem of optimal control for systems with time-varying coefficients, where the time-varying part is of one order of magnitude smaller than the constant part.

## Perturbation Approach

We are concerned here with the case in which the time-varying part of the coefficients in Eq. (5) is of one order of magnitude smaller than the constant part. In this case, we can use a perturbation approach to compute the control gains. To this end, we rewrite Eq. (5) in the state form

$$\dot{\zeta}(t) \cong [A_0 + A_1(t)]\zeta(t) + [B_0 + B_1(t)][F(t) + D(t)] \tag{7}$$

where $\zeta = [\eta^T \; \dot{\eta}^T]^T$ is the pseudomodal state vector and

$$A_0 = \begin{bmatrix} 0 & I \\ -\Lambda_0 & 0 \end{bmatrix} \tag{8a}$$

$$B_0 = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{8b}$$

$$A_1 = \begin{bmatrix} 0 & 0 \\ \overline{M_1}\Lambda_0 - \overline{K_1} & -\overline{G_1} \end{bmatrix} \tag{8c}$$

$$B_1 = \begin{bmatrix} 0 \\ -\overline{M_1} \end{bmatrix} \tag{8d}$$

are coefficient matrices, in which quantities with the subscript 1 are of one order of magnitude smaller than quantities with the subscript 0.

Introducing the notation

$$g = F + D \tag{9}$$

and assuming that the disturbance vector $D$ is constant during the control interval, we can write

$$\dot{g} = \dot{F} \tag{10}$$

The assumption that $D$ is constant requires some explanation. The disturbance vector $D$ depends on the maneuver angular acceleration, the maneuver angular velocity squared, and trigonometric functions of the maneuver angle. Of these, in the case of beam-like appendages, the angular acceleration affects the elastic vibration and all three affect the rigid-body motions. In the case of a bang-bang control law for the maneuver, the angular acceleration is constant over each half of the maneuver. Hence, as far as the elastic vibration is concerned, the assumption of constant disturbance is largely justified. On the other hand, the inertia of the appendages tends to be small compared to the inertia of the rigid platform. Because the disturbance terms involve the inertia of the appendages, the effect on the rigid-body motions is likely to be very small. Hence, treating $D$ as constant can be justified, at least in the case of beam-like structures. This is true in particular when the maneuver is not very rapid. In the case of rapid maneuvers, this assumption must be re-examined. It should be stressed that the assumption is merely for control design, and the time-varying terms in the disturbance are included in the simulation of the closed-loop response.

Inserting Eq. (9) into Eq. (7) and combining with Eq. (10), we obtain the new state equation

$$\dot{w} = \hat{A}w + \hat{B}\dot{F} \tag{11}$$

in which $w = [\zeta^T \; g^T]^T$ is an extended state vector and

$$\hat{A} = \hat{A}_0 + \hat{A}_1(t) \tag{12a}$$

$$\hat{B} = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{12b}$$

are coefficient matrices, where, in Eq. (12a),

$$\hat{A}_0 = \begin{bmatrix} A_0 + A_1(t_f) & B_0 + B_1(t_f) \\ 0 & 0 \end{bmatrix} \tag{13a}$$

$$\hat{A}_1(t) = \begin{bmatrix} A_1(t) - A_1(t_f) & B_1(t) - B_1(t_f) \\ 0 & 0 \end{bmatrix} \tag{13b}$$

in which $\hat{A}_0$ represents the coefficient matrix of the postmaneuver state. We note that $\hat{A}_0$ and $\hat{A}_1(t)$ were defined so that $\hat{A}_1(t)$ reduces to a null matrix at the terminal time $t_f$. This simplifies the boundary conditions of the perturbed solution.

Next, we consider the performance index

$$J = \frac{1}{2}\int_0^T (w^TQw + \dot{F}^TR\dot{F})\,\mathrm{d}t \tag{14}$$

so that the optimal control law is given by

$$\dot{F} = -R^{-1}\hat{B}^TPw \tag{15}$$

where $P$ is the solution of matrix differential Riccati equation[6] (MDRE)

$$-\dot{P} = P\hat{A} + \hat{A}^TP + Q - P\hat{B}R^{-1}\hat{B}^TP \tag{16a}$$

$$P(T) = 0 \tag{16b}$$

Consistent with the perturbation approach, we express $P$ as a sum of matrices of different order of magnitude

$$P = P_0 + P_1 + P_2 + \cdots \tag{17}$$

where the subscript denotes the order of magnitude, in the sense that the higher the value of the subscript the smaller the magnitude order. Inserting Eq. (17) into Eqs. (16), we obtain the zero-order MDRE

$$-\dot{P}_0 = P_0\hat{A}_0 + \hat{A}_0^TP_0 + Q - P_0\hat{B}R^{-1}\hat{B}^TP_0 \tag{18a}$$

$$P_0(T) = 0 \tag{18b}$$

and the higher-order matrix differential Lyapunov equations (MDLEs)

$$-\dot{P}_j = P_j\hat{A}_{0C} + \hat{A}_{0C}^TP_j + \Psi_j \tag{19a}$$

$$P_j(T) = 0, \qquad j = 1, 2, \ldots \tag{19b}$$

where

$$\hat{A}_{0C} = \hat{A}_0 - \hat{B}R^{-1}\hat{B}^TP_0 \tag{20a}$$

denotes a closed-loop coefficient matrix and

$$\Psi_j = P_{j-1}\hat{A}_1 + \hat{A}_1^TP_{j-1} - \sum_{k=1}^{j-1} P_{j-k}\hat{B}R^{-1}\hat{B}^TP_k \qquad j = 1, 2, \ldots \tag{20b}$$

If the final time $T$ approaches infinity and the maneuver ends at time $t_f$, $t_f < T$, then during the postmaneuver period we can use the steady-state solution of the zero-order MDRE because the time-varying coefficients no longer exist after the termination of the maneuver. But, for $T \to \infty$, the zero-order MDRE, Eq. (18a), can be replaced by a matrix algebraic Riccati equation (MARE). Hence, to obtain the zero-order solution, we need only solve

$$P_0\hat{A}_0 + \hat{A}_0^T P_0 + Q - P_0\hat{B}R^{-1}\hat{B}^T P_0 = 0, \qquad t>0 \tag{21}$$

The MDLEs, Eqs. (19), hold only during the maneuver, so that the boundary conditions (19b) must be replaced by $P_j(t_f) = 0$, where $t_f$ indicates the final time of the maneuver. In view of this, the solution of Eqs. (19) can be expressed as

$$P_j(t) = \int_t^{t_f} \exp[\hat{A}_{0C}^T(\tau - t)]\Psi_j(\tau)\exp[\hat{A}_{0C}(\tau - t)]\,\mathrm{d}\tau$$
$$0 \le t \le t_f, \quad j = 1, 2, \ldots \tag{22}$$

as shown in Appendix A. The control law is obtained by adding the solution of the MARE, Eq. (21), to Eq. (22) and inserting the result into Eq. (15).

To evaluate the integral in Eq. (22), it is advisable to resort to discretization in time. Letting $\Delta t_k = t_{k+1} - t_k$, it is shown in Appendix B that Eqs. (22) can be replaced by the backward difference equations

$$P_j(t_k) = \exp[\hat{A}_{0C}^T\Delta t_k]P_j(t_{k+1})\exp[\hat{A}_{0C}\Delta t_k]$$
$$+ \int_0^{\Delta t_k} \exp[\hat{A}_{0C}^T\xi]\Psi_j(t_k + \xi)\exp[\hat{A}_{0C}\xi]\,\mathrm{d}\xi$$
$$k = 0, 1, 2, \ldots, \qquad j = 1, 2, \ldots \tag{23}$$

It is reasonable to assume that the integrand varies linearly over the small time interval $t_k < t < t_{k+1}$, so that Eqs. (23) can be approximated as follows:

$$P_j(t_k) \cong \tfrac{1}{2}\Psi_j(t_k)\Delta t_k + \exp[\hat{A}_{0C}^T\Delta t_k][P_j(t_{k+1})$$
$$+ \tfrac{1}{2}\Psi_j(t_{k+1})\Delta t_k]\exp[\hat{A}_{0C}\Delta t_k]$$
$$k = 0, 1, 2, \ldots, \qquad j = 1, 2, \ldots \tag{24}$$

Equations (24) represent recurrence formulas permitting the computation of $P_j$ backward in time. This task must be carried out prior to the start of the maneuver.

The control law given by Eq. (15) is in the form of a differential equation, so that it does not lend itself to ready implementation. Our object is to use this equation to generate an implementable control law. To this end, we partition the Riccati matrix $P(t)$ as follows:

$$P(t) = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \tag{25}$$

Then, recalling Eq. (12b) and the nature of the extended state vector $\boldsymbol{w}$, Eq. (15) can be rewritten as

$$\dot{\boldsymbol{F}} = -R^{-1}P_{31}\boldsymbol{\eta} - R^{-1}P_{32}\dot{\boldsymbol{\eta}} - R^{-1}P_{33}\boldsymbol{g} \tag{26}$$

Introducing Eq. (9) into Eq. (5), we have simply

$$\boldsymbol{g} = (I + \overline{M_1})\ddot{\boldsymbol{\eta}} + \overline{G_1}\dot{\boldsymbol{\eta}} + (\Lambda_0 + \overline{K_1})\boldsymbol{\eta} \tag{27}$$

so that, inserting Eq. (27) into Eq. (26) and considering a perturbation expansion, we obtain

$$\dot{F} = -\Big[G_i^0 + \sum_j G_i^j(t)\Big]\eta(t) - \Big[G_p^0 + \sum_j G_p^j(t)\Big]\dot{\eta}(t)$$
$$- \Big[G_d^0 + \sum_j G_d^j(t)\Big]\ddot{\eta}(t) \tag{28}$$

where

$$G_i^0 = R^{-1}[P_{31}^0 + P_{33}^0\Lambda_0] \tag{29a}$$

$$G_i^j = R^{-1}[P_{31}^j + P_{33}^j\Lambda_0 + P_{33}^{j-1}\overline{K_1}], \quad j = 1, 2, \ldots \tag{29b}$$

$$G_p^0 = R^{-1}P_{32}^0 \tag{29c}$$

$$G_p^j = R^{-1}[P_{32}^j + P_{32}^{j-1}\overline{G_1}], \quad j = 1, 2, \ldots \tag{29d}$$

$$G_d^0 = R^{-1}P_{33}^0 \tag{29e}$$

$$G_d^j = R^{-1}[P_{33}^j + P_{33}^{j-1}\overline{M_1}], \quad j = 1, 2, \ldots \tag{29f}$$

in which superscripts 0 and $j$ denote time-invariant and small time-varying quantities, respectively. The closed-loop equations are obtained by combining Eqs. (5) and (28).

The response of the closed-loop system can be derived conveniently by casting the closed-loop equations in state form. To this end, we introduce the new state vector $\boldsymbol{\xi} = [\boldsymbol{\eta}^T\ \dot{\boldsymbol{\eta}}^T\ \boldsymbol{F}^T]^T$ and rewrite Eqs. (5) and (28) in the state form

$$A^*(t)\dot{\xi}(t) = B^*(t)\xi(t) + D^*D(t) \tag{30}$$

where

$$A^*(t) = \begin{bmatrix} I & 0 & 0 \\ 0 & I + \overline{M_1} & 0 \\ 0 & G_d^0 + \sum_j G_d^j & I \end{bmatrix} \tag{31a}$$

$$B^*(t) = \begin{bmatrix} 0 & I & 0 \\ -\Lambda_0 - \overline{K_1} & -\overline{G_1} & I \\ -G_i^0 - \sum_j G_i^j & -G_p^0 - \sum_j G_p^j & 0 \end{bmatrix} \tag{31b}$$

$$D^* = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \tag{31c}$$

in which $F(0)$ was chosen as zero arbitrarily. Moreover, we assumed that the maneuver starts from rest, so that $\eta(0)$ and $\dot{\eta}(0)$ are zero as well. Integration of Eq. (30) can be carried out in discrete time.

## Stability Analysis

For future reference, we would like to examine the stability characteristics of the closed-loop system. To this end, we introduce Eq. (15) into Eq. (11) and obtain the closed-loop equation

$$\dot{w} = \hat{A}_C w \tag{32}$$

where

$$\hat{A}_C = \hat{A} - \hat{B}R^{-1}\hat{B}^T P \tag{33}$$

is the closed-loop coefficient matrix. To investigate the stability of the system described by Eq. (32), we consider Lyapunov's second method. The second stability theorem of Lyapunov is concerned with the asymptotic stability of a system in the neighborhood of the origin and it reads as follows: If there exists for the system (32) a positive definite function $V(\boldsymbol{w})$

whose total time derivative $\dot{V}(w)$ is negative definite along every trajectory of Eq. (32), then the trivial solution is asymptotically stable.

The matrix $P(t)$ that satisfies Eq. (16a) is positive definite, so that

$$V = w^T P w > 0 \quad \text{for all} \quad w \neq 0 \quad \text{and for all} \quad t \tag{34}$$

Hence, the scalar function $V$ defined by Eq. (34) can be regarded as a candidate for a Lyapunov function. Next, consider Eq. (32) and write

$$\dot{V} = w^T(\dot{P} + P\hat{A}_C + \hat{A}_C^T P)w = w^T H w \tag{35}$$

where

$$H = \dot{P} + \hat{A}_C^T P + P\hat{A}_C \tag{36}$$

or, using Eq. (16a),

$$H = -Q - P\hat{B}R^{-1}\hat{B}^T P \tag{37}$$

The stability of the system depends on the negative definiteness of $H$. If the eigenvalues of $H$ are all negative, the system is guaranteed to be stable. From Eq. (37), the negative definiteness of $H$ can be easily verified, so that stability is guaranteed. In the case of the perturbation approach, Eq. (37) becomes

$$H = -Q - (P_0 + P_1 + P_2 + \cdots)\hat{B}R^{-1}\hat{B}^T(P_0 + P_1 + P_2 \cdots) \tag{38}$$

so that once again stability is guaranteed.

## Adiabatic Approximation

Another approach to control design for time-varying systems, referred to as "adiabatic approximation," is suggested in Ref. 5. If the rate of variation of the parameters is slow relative to the closed-loop dynamic response, then one can design the control gains under the assumption that the system is time invariant and schedule the gains as functions of time. The gains are determined in Ref. 5 by solving a matrix algebraic Riccati equation at various instants of time. This raises questions of stability, which are answered in Ref. 5 by giving sufficient conditions for asymptotic stability. If we use the concept of adiabatic approximation to the perturbed Riccati equation, the adiabatic solution for the time-varying part can be obtained by solving the MDLE, Eq. (19a), by letting $\dot{P}_j = 0$ for each instant of time. The resulting matrix algebraic Lyapunov equations are

$$P_j(t)\hat{A}_{0C} + \hat{A}_{0C}^T P_j(t) + \Psi_j = 0, \qquad j = 1, 2, \ldots \tag{39}$$

The advantage of this solution is that the time-varying gain matrix can be calculated at each instant of time without solving the differential equation. The gains can be calculated prior to any maneuver, thus saving real-time computations. However, stability must be checked a priori.

To check the stability characteristics of the control designed by the adiabatic approximations, we use the same approach as for the perturbation approach. Hence, following the same procedure as for the perturbation approach, it can be verified that in the case of the adiabatic approximation we obtain

$$H_{\text{adia}} = \sum_j \dot{P}_j(t) + H_{\text{pert}} \tag{40}$$

The negative definiteness of $H_{\text{adia}}$ depends on the contribution of $\dot{P}_j$ to $H_{\text{adia}}$. Although $P_j$ may be small, the time-derivative $\dot{P}_j$ can be large if the time-varying terms change abruptly. This can happen during bang-bang maneuvering, for which there is a rapid change from acceleration to deceleration at half the maneuver period.

## Numerical Example

As a numerical example, we consider the planar model shown in Fig. 2. In the two-dimensional case, the configuration vector in Eq. (1) can be written as $x = [R_y \, R_z \, \theta \, q_e^T]^T$ where $R_y$ and $R_z$ represent the translations in the $y$ and $z$ directions, $\theta$ represents the angular motion of the main rigid body, and $q_e$ is a vector of generalized coordinates associated with the elastic motion of the appendage. The coefficient matrices entering into Eq. (1) are as follows:

$$M_0 = \begin{bmatrix} m_t & 0 & 0 & 0 \\ 0 & m_t & S_t & \bar{\Phi}_e \\ 0 & S_t & I_t & \tilde{\Phi}_e + r_{0e}\bar{\Phi}_e \\ 0 & \bar{\Phi}_e^T & \tilde{\Phi}_e^T + r_{0e}\bar{\Phi}_e^T & M_e \end{bmatrix} \tag{41a}$$

$$K_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_e \end{bmatrix} \tag{41b}$$

are the constant part of the coefficient matrices and

$$M_1 = -s\beta M_s - (1 - c\beta)M_c \tag{42a}$$

$$K_1 = (\dot{\beta}^2 s\beta - \ddot{\beta}c\beta)K_{b1} - (\dot{\beta}^2 c\beta + \ddot{\beta}s\beta)K_{b2} - \dot{\beta}^2 K_{b3} \tag{42b}$$

$$G_1 = -2\dot{\beta}(s\beta G_s + c\beta G_c) \tag{42c}$$

are the time-varying parts, where $\beta$ represents the angle between the flexible body and the rigid body, in which $s\beta$ and $c\beta$ denote $\sin\beta$ and $\cos\beta$, respectively. Moreover,

$$M_s = \begin{bmatrix} 0 & 0 & S_e & \bar{\Phi}_e \\ 0 & 0 & 0 & 0 \\ S_e & 0 & 0 & 0 \\ \bar{\Phi}_e^T & 0 & 0 & 0 \end{bmatrix} \tag{43a}$$

$$M_c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & S_e & \bar{\Phi}_e \\ 0 & S_e & 2S_e r_{0e} & r_{0e}\bar{\Phi}_e \\ 0 & \bar{\Phi}_e^T & r_{0e}\bar{\Phi}_e^T & 0 \end{bmatrix} \tag{43b}$$

$$K_{b1} = \begin{bmatrix} 0 & 0 & S_e & \bar{\Phi}_e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{43c}$$

$$K_{b2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & S_e & \bar{\Phi}_e \\ 0 & 0 & 0 & r_{0e}\bar{\Phi}_e \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{43d}$$

$$K_{b3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_e \end{bmatrix} \tag{43e}$$

$$G_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & S_e & \bar{\Phi}_e \\ 0 & 0 & r_{0e}S_e & r_{0e}\bar{\Phi}_e \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{43f}$$

$$G_c = \begin{bmatrix} 0 & 0 & S_e & \bar{\Phi}_e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{43g}$$

Finally, $f(t)$ and $d(t)$ represent resultant control forces and disturbances, respectively, and they are given by

$$f = [F_y \quad F_z \quad M_x \quad Q_e^T]^T \tag{44a}$$

and

$$d = \begin{bmatrix} S_e(\dot{\beta}^2 c\beta + \ddot{\beta} s\beta) \\ S_e(\dot{\beta}^2 s\beta - \ddot{\beta} c\beta) \\ -I_e\ddot{\beta} + S_e r_{0e}(\dot{\beta}^2 s\beta - \ddot{\beta} c\beta) \\ -\tilde{\Phi}_e^T \ddot{\beta} \end{bmatrix} \tag{44b}$$

The various terms entering into Eqs. (41), (43), and (44b) are defined as

$$m_t = m_r + m_e \tag{45a}$$

$$S_t = S_e + m_e r_{0e} \tag{45b}$$

$$I_t = I_r + I_e + m_e r_{0e}^2 + 2r_{0e}S_e \tag{45c}$$

$$S_e = \int_{D_e} \rho_e r_e \, dD_e \tag{45d}$$

$$I_r = \int_{D_r} \rho_r(r_y^2 + r_z^2) \, dD_r \tag{45e}$$

$$I_e = \int_{D_e} \rho_e r_e^2 \, dD_e \tag{45f}$$

$$\bar{\Phi}_e = \int_{D_e} \rho_e \Phi_e \, dD_e \tag{45g}$$

$$\tilde{\Phi}_e = \int_{D_e} \rho_e r_e \Phi_e \, dD_e \tag{45h}$$

$$M_e = \int_{D_e} \rho_e \Phi_e^T \Phi_e \, dD_e \tag{45i}$$

$$K_e = [\Phi_e, \Phi_e] \tag{45j}$$

in which $\rho_r$ and $\rho_e$ represent the mass density for the rigid body and flexible body; $r_{0e}$ is the radius vector from 0 to $e$, where the point 0 is taken to be the center of mass of the rigid body; $r_e$ is the position vector of a nominal point in the undeformed appendage relative to $x_e y_e z_e$; $m_r$ and $m_e$ represent the mass of the rigid and flexible bodies; $S_e$ represents the first mass moment of inertia of the flexible body about point $e$; $I_r$ and $I_e$ are the mass moments of inertia of the rigid body and flexible body; and [ , ] represents an energy inner product.[7] The vibration was represented by five admissible functions, i.e., $\Phi_e = [\phi_1, \phi_2, \ldots, \phi_5]$, where each admissible function has the expression

$$\phi_j = -(\cos\gamma_j y_1 - \cosh\gamma_j y_1) + C_j(\sin\gamma_j y_1 - \sinh\gamma_j y_1)$$

$$j = 1, 2, \ldots, 5 \tag{46}$$

which are recognized as cantilever shape functions.[7] The coefficients in Eqs. (46) have the values $C_j = 0.7341$, 1.0185, 0.9992, 1, 1, and the arguments of the trigonometric and hyperbolic functions can be obtained from $\gamma_j l_e = 1.8751$, 4.6941, 7.8548, 10.9955, 14.1372, where $l_e$ is the length of the beam.

The maneuver of the appendage relative to the platform was carried out by means of a bang-bang control law for the angular acceleration. Hence, we have

$$\ddot{\beta} = c, \quad \dot{\beta} = ct, \quad \beta = \tfrac{1}{2}ct^2 \text{ for } t \le t_f/2 \tag{47a}$$

$$\ddot{\beta} = -c, \quad \dot{\beta} = -c(t-t_f), \quad \beta = -\tfrac{1}{2}c(t-t_f)^2 + \tfrac{1}{4}ct_f^2$$

$$\text{for} \quad t_f/2 \le t \le t_f \tag{47b}$$

where $c = 4\beta_f/t_f^2$, in which $\beta_f$ and $t_f$ represent the final maneuver angle and time.

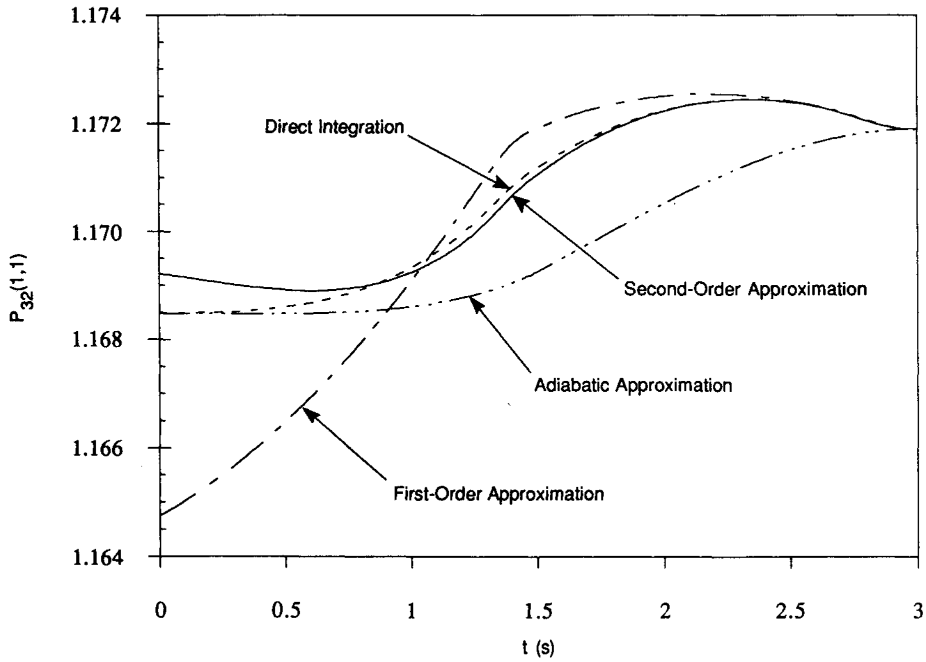


Fig. 3 Time history of $P_{32}(1,1)$.

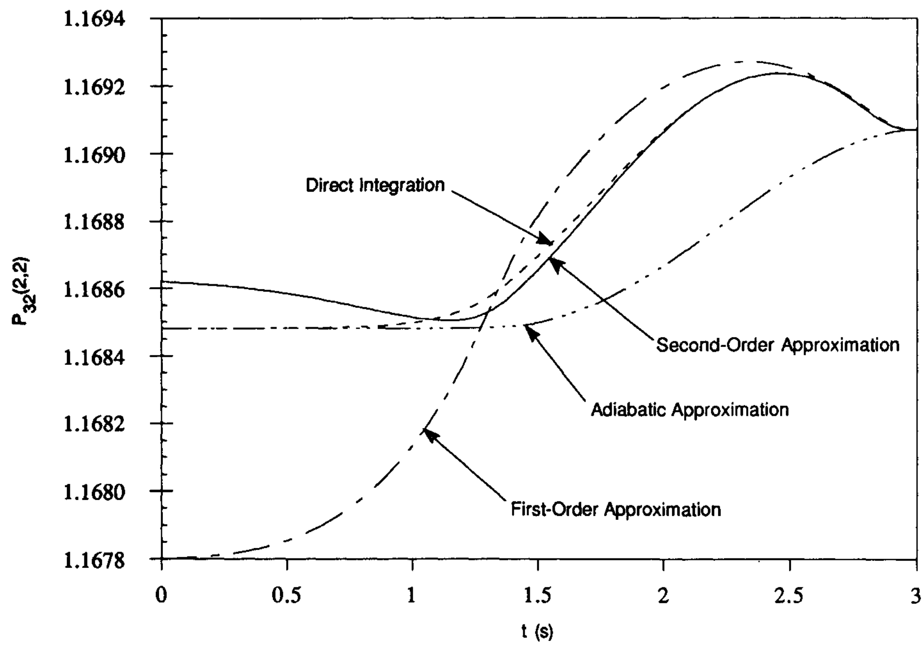


**Fig. 4 Time history of $P_{32}(2,2)$.**

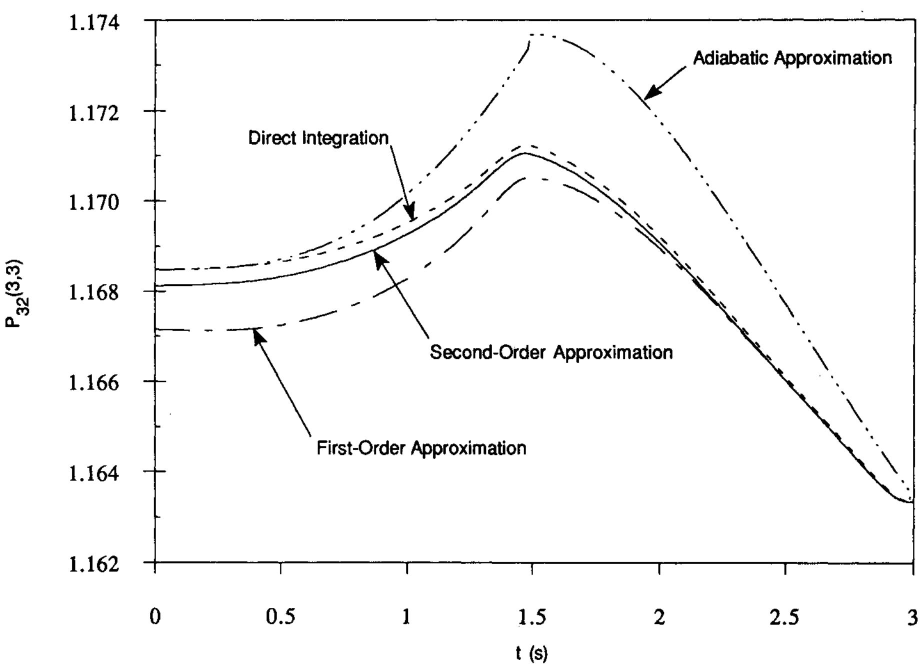


**Fig. 5 Time history of $P_{32}(3,3)$.**

The numerical data used are as follows:

$$m_r = 10 \text{ kg}, \qquad m_e = 1 \text{ kg}$$

$$S_e = 2.5 \text{ kg} \cdot \text{m}$$

$$I_r = 20.0 \text{ kg} \cdot \text{m}^2, \qquad I_e = 8.33 \text{ kg} \cdot \text{m}^2$$

$$r_{0e} = 2 \text{ m}, \quad l_e = 5 \text{ m}, \quad EI_e = 600 \text{ N} \cdot \text{m}^2$$

$$\bar{\Phi}_e = m_e[0.783\ 0.434\ 0.254\ 0.182\ 0.141] \text{ kg}$$

$$\tilde{\Phi}_e = m_e l_e[0.569\ 0.091\ 0.032\ 0.017\ 0.010] \text{ kg} \cdot \text{m}$$

$$M_e = m_e \delta_{ij}, \qquad K_e = \frac{EI_e}{l_e^3}(\gamma_i l_e)^2(\gamma_j l_e)^2 \delta_{ij} + K_{eg}$$

$$\beta_f = 45 \text{ deg}, \qquad t_f = 3 \ s$$

where $K_{eg}$ represents the geometric stiffening effects. These effects can be attributed to inertial forces acting over a shortening of the projection due to bending displacements[8] and can be significant for rapid maneuvers. In the present example, they are negligibly small.

Figures 3–6 show some of the entries of the Riccati matrix obtained by direct numerical integration, adiabatic approximation, first-order perturbation method, and second-order perturbation method. Because the direct integration of the matrix differential Riccati equation corresponding to a model including five elastic degrees of freedom was not possible due to numerical difficulties, comparison of Riccati solutions was made based on a model including only one elastic degree of freedom. As can be concluded from the figures, the solutions obtained by the adiabatic approximation differ to some extent from the solutions obtained by direct numerical integration,

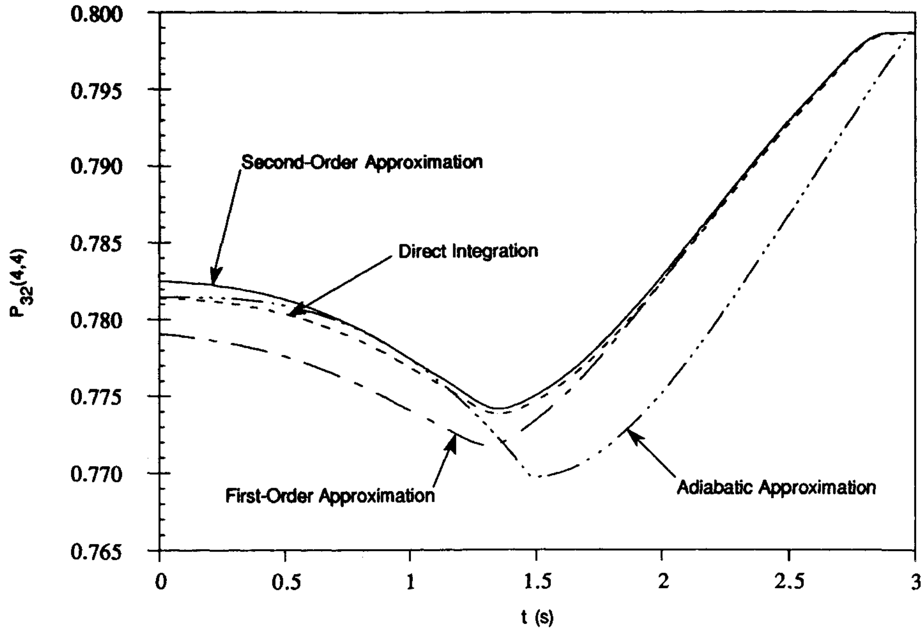


Fig. 6 Time history of $P_{32}(4,4)$.

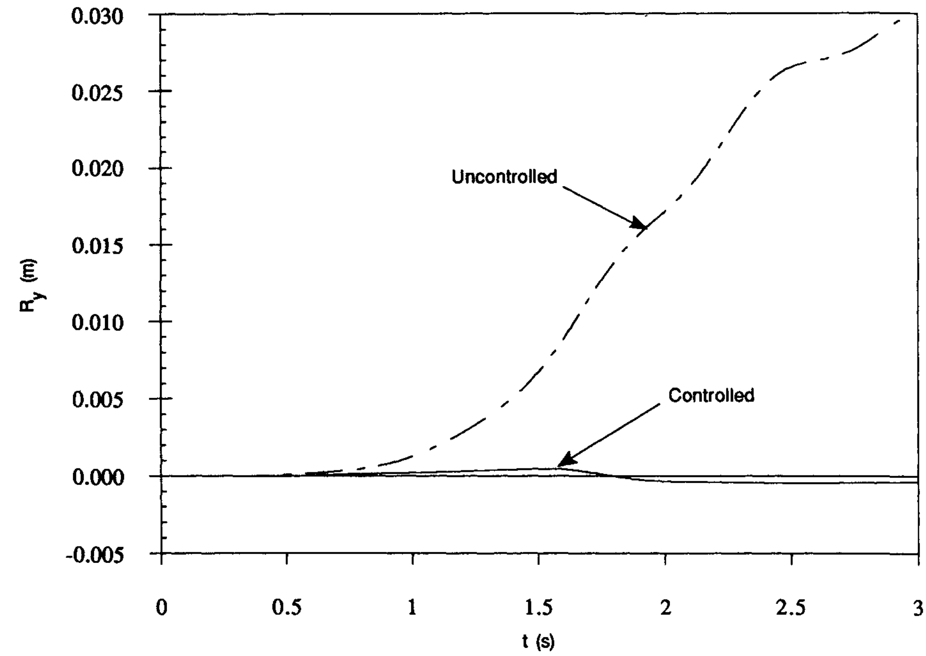


Fig. 7 Translation of the rigid platform in the $y$ direction.

and the same can be said about the first-order perturbation solutions. On the other hand, the second-order perturbation solutions are very close to the solutions obtained by direct numerical integration. Note that the entries represent the diagonal elements of $P_{32}$; the off-diagonal elements of $P_{32}$ are several orders of magnitude smaller. Note also that the scale of the plots may be misleading and in fact the solutions are closer together than they seem.

Figures 7–10 show the uncontrolled and controlled rigid-body and elastic displacements obtained by the second-order perturbation method for a model including five elastic degrees of freedom. The coefficient matrices in the performance measure were as follows:

$$Q = \begin{bmatrix} 100I_{16} & 0 \\ 0 & I_8 \end{bmatrix}, \qquad R = 0.01I_8$$

Note that computation of the control gains by direct integration ran into numerical difficulties, so that no comparison was possible. Finally, Fig. 11 shows the time history of the control forces and control torque for the rigid-body displacements, and Fig. 12 shows the time-history of the generalized controls associated with the elastic degrees of freedom.

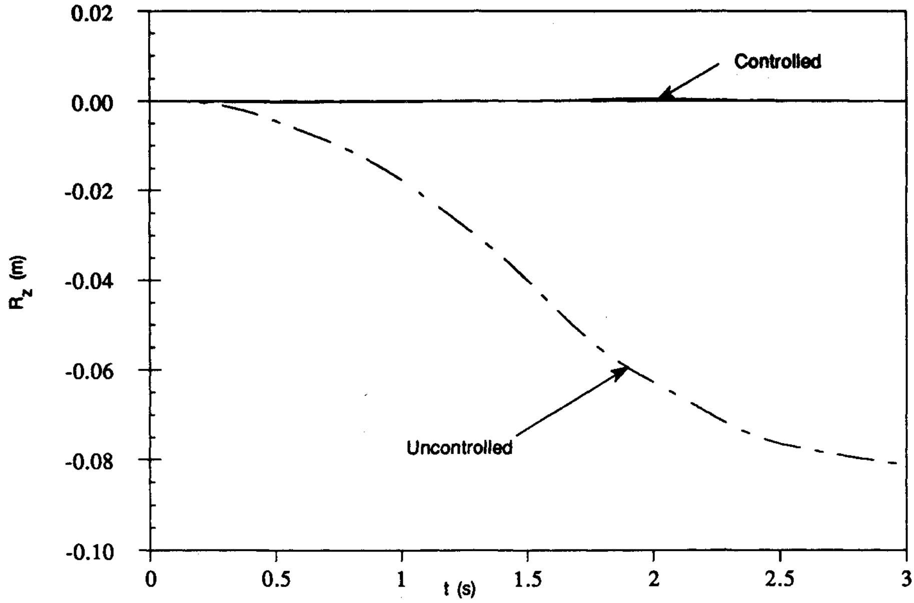


Fig. 8 Translation of the rigid platform in the $z$ direction.

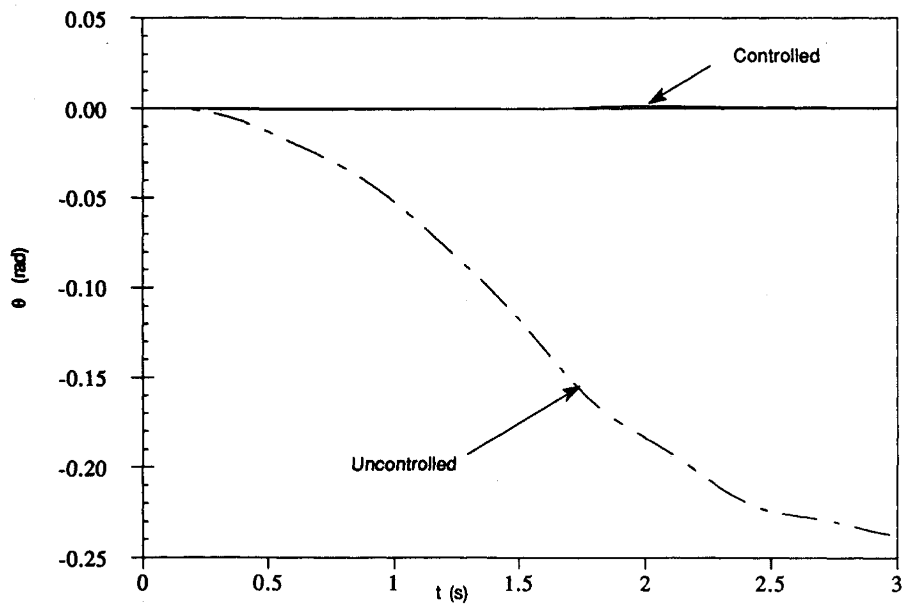


Fig. 9 Rotation of the rigid platform about the $x$ axis.

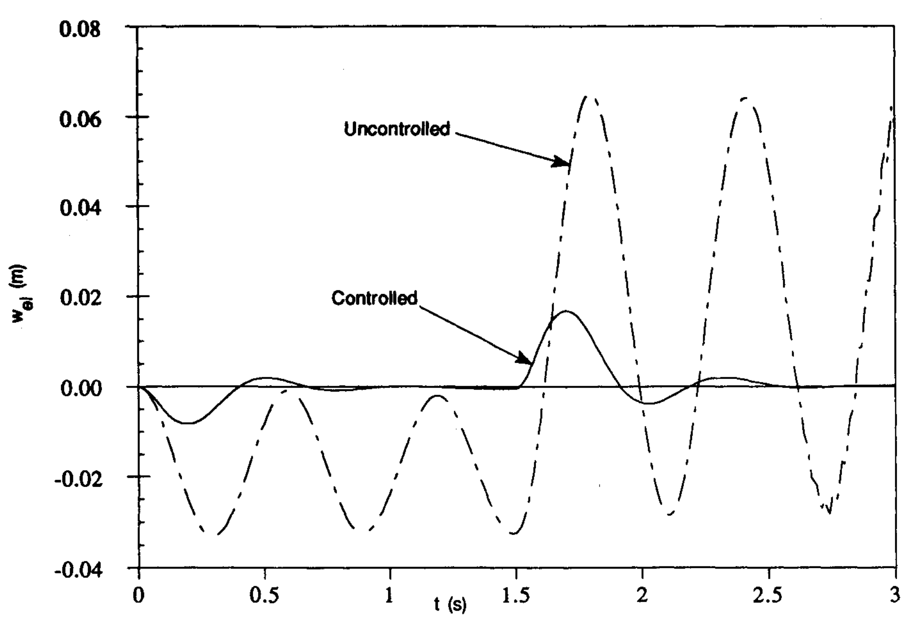


Fig. 10 Elastic displacement of the tip of the flexible appendage.

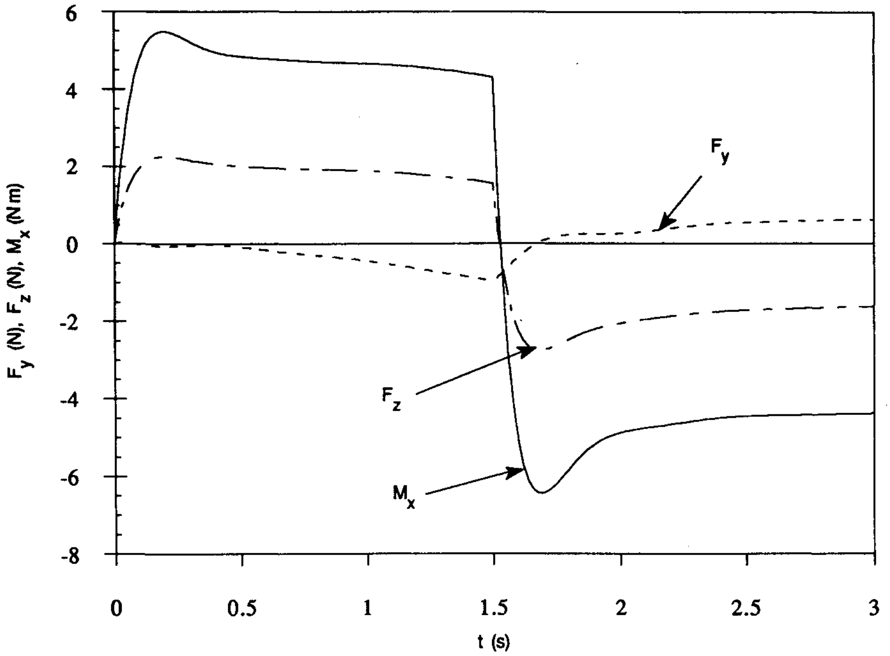


**Fig. 11 Control forces and torque for the rigid-body motions.**

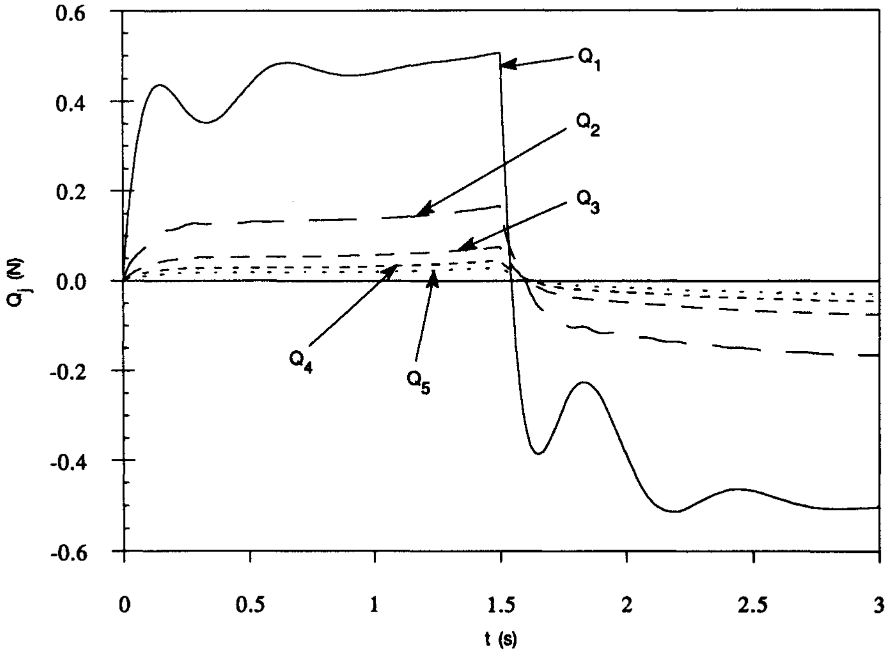


**Fig. 12 Generalized control forces for the elastic degrees of freedom.**

## Summary and Conclusions

The equations governing the motion of a flexible spacecraft with time-varying configuration, such as in the case of maneuvering flexible appendages, are characterized by time-varying coefficients and persistent disturbances. A perturbation technique is developed here for the design of optimal controls for the case in which the time-varying part of the coefficient is small compared with the constant part. According to this perturbation technique, the determination of the control gains is divided into two parts, a time-invariant part obtained by solving a matrix algebraic Riccati equation and a time-varying part obtained by solving matrix differential Lyapunov equations. The resulting closed-loop equations represent a set of equations with small time-varying coefficients, which can be treated by extending the state to include the control vector. Integration of the resulting state equations can be carried out in discrete time. A numerical example in which a flexible appendage is reoriented relative to a platform stabilized in an inertial space demonstrates the effectiveness of the method.

## Appendix A: Solution of the Matrix Differential Lyapunov Equations

Introducing the change of variables $t = t_f - \tau$, Eq. (19a) becomes

$$P_j' = P_j\hat{A}_{0C} + \hat{A}_{0C}^T P_j + \Psi_j(t_f - \tau) \tag{A1}$$

where prime denotes $d/d\tau$. Moreover, boundary condition (19b) becomes $P_j(0) = 0$. Multiplying Eq. (A1) on the left by $S_1(\tau)$ and on the right by $S_2(\tau)$, we obtain

$$S_1 P_j' S_2 = S_1 P_j \hat{A}_{0C} S_2 + S_1 \hat{A}_{0C}^T P_j S_2 + S_1 \Psi_j(t_f - \tau) S_2 \tag{A2}$$

Next, we consider

$$\frac{d}{d\tau}(S_1 P_j S_2) = S_1' P_j S_2 + S_1 P_j' S_2 + S_1 P_j S_2' \tag{A3}$$

so that Eq. (A2) can be rewritten as

$$\frac{d}{d\tau}(S_1 P_j S_2) - S_1 P_j (S_2' + \hat{A}_{0C} S_2) - (S_1' + S_1 \hat{A}_{0C}^T) P_j S_2 = S_1 \Psi_j(t_f - \tau) S_2 \tag{A4}$$

Assuming that $S_1$ and $S_2$ satisfy

$$S_1' + S_1 \hat{A}_{0C}^T = 0 \tag{A5a}$$

$$S_2' + \hat{A}_{0C} S_2 = 0 \tag{A5b}$$

Eq. (A4) reduces to

$$\frac{d}{d\tau}(S_1 P_j S_2) = S_1 \Psi_j(t_f - \tau) S_2 \tag{A6}$$

Integrating Eq. (A6), we obtain

$$S_1(\sigma) P_j(\sigma) S_2(\sigma) = \int_0^\sigma S_1(\tau) \Psi_j(t_f - \tau) S_2(\tau)\, d\tau \tag{A7}$$

Equation (A7) yields

$$P_j(\sigma) = S_1^{-1}(\sigma) \int_0^\sigma S_1(\tau) \Psi_j(t_f - \tau) S_2(\tau)\, d\tau S_2^{-1}(\sigma) \tag{A8}$$

Equations (A5) have the solution

$$S_1(\tau) = \exp(-\hat{A}_{0C}^T \tau) S_1(0) \tag{A9a}$$

$$S_2(\tau) = \exp(-\hat{A}_{0C}^T \tau) S_2(0) \tag{A9b}$$

so that, inserting Eqs. (A9) into Eq. (A8) we obtain

$$P_j(\sigma) = \int_0^\sigma \exp[\hat{A}_{0C}^T(\sigma - \tau)] \Psi_j(t_f - \tau) \exp[\hat{A}_{0C}(\sigma - \tau)]\, d\tau \tag{A10}$$

Introducing the changes of variables $\gamma = t_f - \tau$ and $\sigma = t_f - t$, we can rewrite Eq. (A10) as

$$P_j(t) = \int_t^{tf} \exp[\hat{A}_{0C}^T(\gamma - t)] \Psi_j(\gamma) \exp[\hat{A}_{0C}(\gamma - t)]\, d\tau \tag{A11}$$

Finally, replacing $\gamma$ by $\tau$, we obtain Eq. (22).

## Appendix B: Discrete-Time Solution of the Matrix Differential Lyapunov Equations

The object is to discretize Eq. (22) in time. To this end, we let $t = t_{k+1}$ in Eq. (22) and write

$$P_j(t_{k+1}) = \int_{t_{k+1}}^{tf} \exp[\hat{A}_{0C}^T(\tau - t_{k+1})] \Psi_j(\tau) \times \exp[\hat{A}_{0C}(\tau - t_{k+1})]\, d\tau \tag{B1}$$

Then, letting $t = t_k = t_{k+1} - \Delta t_k$ in Eq. (22) and considering Eq. (B1), we obtain

$$\begin{aligned} P_j(t_k) &= \int_{t_k}^{tf} \exp[\hat{A}_{0C}^T(\tau - t_k)] \Psi_j(\tau) \exp[\hat{A}_{0C}(\tau - t_k)]\, d\tau \\ &= \int_{t_{k+1}}^{tf} \exp[\hat{A}_{0C}^T(\tau - t_k)] \Psi_j(\tau) \exp[\hat{A}_{0C}(\tau - t_k)]\, d\tau \\ &\quad + \int_{t_k}^{t_{k+1}} \exp[\hat{A}_{0C}^T(\tau - t_k)] \Psi_j(\tau) \exp[\hat{A}_{0C}(\tau - t_k)]\, d\tau \\ &= \exp(\hat{A}_{0C}^T \Delta t_k) P_j(t_{k+1}) \exp(\hat{A}_{0C} \Delta t_k) \\ &\quad + \int_{t_k}^{t_{k+1}} \exp[\hat{A}_{0C}^T(\tau - t_k)] \Psi_j(\tau) \exp[\hat{A}_{0C}(\tau - t_k)]\, d\tau \\ &= \exp(\hat{A}_{0C}^T \Delta t_k) P_j(t_{k+1}) \exp(\hat{A}_{0C} \Delta t_k) \\ &\quad + \int_0^{\Delta tk} \exp(\hat{A}_{0C}^T \xi) \Psi_j(t_k + \xi) \exp(\hat{A}_{0C} \xi)\, d\xi \end{aligned} \tag{B2}$$

where we introduced the change of variables $\tau = t_k + \xi$ in the integral.

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## References

[1]Meirovitch, L., and Kwak, M. W., "Dynamics and Control of a Spacecraft with Retargeting Flexible Antennas," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 2, 1990, pp. 241–248.

[2]Johnson, C. D., "Optimal Control of the Linear Regulator with Constant Disturbances," *IEEE Transactions on Automatic Control,* Vol. AC-13, No. 4, 1968, pp. 416–421.

[3]Anderson, B. D. O., and Moore, J. B., *Optimal Control,* Prentice-Hall, Englewood Cliffs, NJ, 1990, Sec. 9.3.

[4]Meirovitch, L., and Kwak, M. K., "Control of a Spacecraft with Multi-Targeted Flexible Antennas," *Journal of the Astronautical Sciences,* Vol. 38, No. 2, 1990, pp. 187–199.

[5]Friedland, B., Richman, J., and Williams, D. E., "On the 'Adiabatic Approximation' for Design and Control Laws for Linear, Time-Varying Systems," *IEEE Transactions on Automatic Control,* Vol. AC-32, No. 1, 1987, pp. 62,63.

[6]Meirovitch, L., *Dynamics and Control of Structures,* Wiley-Interscience, New York, 1990, Sec. 6.4.

[7]Meirovitch, L., *Computational Methods in Structural Dynamics,* Sijthoff & Noordhoff, Alphen aan der Rijn, The Netherlands, 1980, Sec. 7.4.

[8]Meirovitch, L., "Hybrid State Equations of Motion for Flexible Bodies in Terms of Quasi-Coordinates," *Journal of Guidance, Control, and Dynamics,* Vol. 14, No. 5, 1991, pp. 1008–1013.